

HIGHER-ORDER QUANTIZATION ON A LIE GROUP *

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Abstract

In this paper we are mainly concerned with the study of polarizations (in general of higher-order type) on a connected Lie group with a $U(1)$ -principal bundle structure. The representation technique used here is formulated on the basis of a group quantization formalism previously introduced which generalizes the Kostant-Kirillov co-adjoint orbits method for connected Lie groups and the Borel-Weyl-Bott representation algorithm for semisimple groups. We illustrate the fundamentals of the group approach with the help of some examples like the abelian group R^k and the semisimple group $SU(2)$, and the use of higher-order polarizations with the harmonic oscillator group and the Schrödinger group, the last one constituting the simplest example of an anomalous group. Also, examples of infinite-dimensional anomalous groups are briefly considered.

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1 Introduction

This paper is devoted mainly to the concept of higher-order polarization on a Lie group as a powerful tool in search of irreducibility of the representations of the group, and/or the irreducibility of quantizations in a group-theoretic quantization approach, in those *anomalous* cases where standard (geometric) methods do not succeed. The co-adjoint orbit method has proven to fail in quantizing, for instance, non-Kähler orbits of certain Lie groups or orbits without any invariant (first-order or standard) polarization. Also, the configuration-space image of quantization of rather elementary physical systems cannot be obtained in a natural way via the Geometric Quantization technique, but it is obtained easily with the aid of higher-order polarizations.

Higher-order polarizations are naturally defined on the (left) enveloping algebra of a Lie group as higher-order differential operators acting on complex functions on the group manifold. Since higher-order polarizations do not coincide, in general, with the enveloping algebra of any first-order polarization subalgebra, the space of wave functions is not necessarily associated with any particular classical configuration space, thus leading to a *breakdown of the notion of classical limit* for those systems which are anomalous (see Sec. 4). It should be remarked that higher-order polarizations excludes non-differential operators, such as discrete transformations (like parity or charge conjugation), so that a higher-order quantization on a group does not guarantee a full reduction of the representation in the general case; it can lead, for instance, to the direct sum of irreducible representations which are only distinguished by the eigenvalues of a discrete operator commuting with the representation (see the example of Sec. 4). A generalization of the concept of polarization including this sort of non-local operators is beyond the scope of the present paper, but will be studied elsewhere.

The first general attempt to look at the problem of quantization in terms of groups of transformations was done by L. van Hove [1], who tried to associate a group of unitary transformations with the group of canonical transformations on the phase space $M = R^{2n}$. The main point in van Hove's approach was to consider a Hilbert space of complex valued functions on the phase space M , and to implement on these wave functions the action of canonical transformations as unitary transformations according to

$$1) \quad \phi \rightarrow U_\phi \tag{1}$$

$$2) \quad (U_\phi f)(p, q) = (e^{-iS_\phi} f)(P, Q),$$

where S_ϕ is the generating function of the canonical transformation $\phi : (P, Q) \rightarrow (p, q)$:

$$p_i dq^i - P_i dQ^i = dS_\phi. \quad (2)$$

When the transformation ϕ is a one-parameter group of transformations an infinitesimal relation for the generating function can be associated:

$$L_X(p_i dq^i) = dS_X, \quad (3)$$

where X is the infinitesimal generator of the canonical one-parameter group ϕ_t . It should be stressed that because of the phase factor in front of f in (1), this action does not respect the ring structure (it is not an automorphism) of $\mathcal{F}(M)$, indeed $U_\phi(f_1 \cdot f_2) \neq U_\phi(f_1) \cdot U_\phi(f_2)$.

Furthermore, in the known duality relation between a manifold M and the ring of (germs of) functions on it, $\mathcal{F}(M)$, a one-to-one correspondence between automorphisms of $\mathcal{F}(M)$ and diffeomorphisms on M can be established. Therefore, if U_ϕ were an automorphism of the ring of functions when ϕ is a one-parameter group, a vector field could be associated with it, and this would coincide with the infinitesimal generator of U_ϕ as from the Stone-von Neumann theorem [2], after a measure would have been introduced associated with the canonical symplectic structure.

It should be realized that U_ϕ is a ring automorphism (up to a redefinition to absorb a global phase) only if $dS_\phi = 0$ or, in equivalent terms, $dS_X = 0$. To achieve this result we may enlarge the carrier space M to $\tilde{M} = M \times U(1)$. We use π to denote the projection $\pi : \tilde{M} \rightarrow M$, and define $\tilde{\theta} = \pi^*(p_i dq^i) - ds$, s being the extra coordinate, and $\tilde{X} = X + S_X \frac{\partial}{\partial s}$. Then

$$L_{\tilde{X}} \tilde{\theta} = \pi^*(L_X(p_i dq^i)) - dS_X = 0. \quad (4)$$

Thus, the lift of our one-parameter group of transformations U_ϕ to the extended space would be a ring automorphism and, therefore, associated with an infinitesimal generator which is a vector field on \tilde{M} . This is a possible motivation for introducing a $U(1)$ “extension” of our symplectic manifold M . Geometric Quantization [3, 4] generalizes this construction to any symplectic manifold M with symplectic 2-form of integer cohomology class. In the general case, the principal fibration $\tilde{M} \rightarrow M$ with structure group $U(1)$ is not necessarily trivial.

After this motivation, in Sec. 2 the group approach to quantization is introduced, illustrated with the examples of the quantization of the abelian group R^k , leading to the Heisenberg-Weyl commutation relations, and the quantization of the semisimple group $SU(2)$, in Sec. 3. The notion of (algebraic) anomaly is discussed and the particular cases of the Schrödinger and Virasoro groups are considered in Sec. 4. In Sec. 5 we provide a precise definition of higher-order polarizations as well as examples of the use of higher-order polarizations for the case of the quantization of the (non-anomalous) harmonic oscillator group in configuration space and the Schrödinger group. Finally, Sec. 6 is devoted to comments and outlooks. The definitions of both first-order and higher-order polarizations given in this paper generalize in some respects the ones introduced in previous papers (see [5, 6]).

2 Group Approach to Quantization (GAQ)

Now we are no longer dealing necessarily with a symplectic manifold, but rather with a Lie group $\tilde{M} = \tilde{G}$. The discussion in the Introduction represents the motivation for starting with a Lie group which is a $U(1)$ -bundle with base manifold M , in particular, with a central extension \tilde{G} of $G = M$. From the physical point of view, this is traced back to the fact that the Hilbert space of physical states is made out not of functions, but of rays, that is, equivalence classes of functions differing by a phase. This allows for representations of G which are projective representations, and these are true representations of certain central extensions \tilde{G} of G .

We shall restrict ourselves to the study of groups G which are connected and simply connected. This is by no means a drawback of our formalism, since for a non-simply connected group G , if \tilde{G} is the universal covering group and $p : \tilde{G} \rightarrow G$ is the canonical projection, a projective unitary irreducible representation U of \tilde{G} is also a projective unitary irreducible representation of G if and only if $U(\bar{g}) = e^{i\alpha} I, \forall \bar{g} \in \ker p$. In this way, we can obtain easily the representations of G if we know the representations of its universal covering group \tilde{G} .

The connection 1-form $\tilde{\theta} \equiv \Theta$ on \tilde{G} will be naturally selected (see below) among the components of the left-invariant, Lie Algebra valued, Maurer-Cartan 1-form, in such a way that $\Theta(\tilde{X}_0) = 1$ and $L_{\tilde{X}_0} \Theta = 0$, where \tilde{X}_0 is the vertical (or fundamental) vector field. Θ will play the role of a connection

1-form associated with the $U(1)$ -bundle structure.

Therefore, let us suppose that G is a connected and simply connected Lie group and \tilde{G} is a central extension of G by $U(1)$. On \tilde{G} we shall consider the space of complex functions Ψ satisfying the $U(1)$ -equivariance condition, i.e. $\Psi(\zeta * g) = \zeta \Psi(g)$, on which the left translations of \tilde{G} (generated by right-invariant vector fields \tilde{X}^R) realize a representation of the group which is in general reducible. Fortunately, right translations (generated by left-invariant vector fields \tilde{X}^L) on any Lie group do commute with left ones, so that polarization subgroups $G_{\mathcal{P}}$ can be sought (see below) to perform a (maximal) reduction of the representation through polarization conditions, defining a subspace of wavefunctions \mathcal{H} satisfying $\Psi(g * G_{\mathcal{P}}) = \Psi(g)$ (or $L_{\tilde{X}^L} \Psi = 0$, $\forall \tilde{X}^L \in \mathcal{P}$, where \mathcal{P} is the Lie algebra of $G_{\mathcal{P}}$). For a rather wide class of groups, polarization subalgebras in the Lie algebra suffice to achieve the full reduction, whereas groups called anomalous (precisely those for which the Kirillov-Kostant method is not appropriate, see Sec. 4) require generalized or higher-order polarization subalgebras of the enveloping algebra of \tilde{G} , and even so, some special representations can be only reduced to a direct sum of irreducible representations distinguished by the eigenvalues of a discrete operator not belonging to the group (see Sec. 5).

Finally, the Hilbert space structure on the space of wave functions is provided by the invariant Haar measure constituted by the exterior product of the components of the left-invariant canonical 1-form, $\Omega^L \equiv \theta^{Lg^1} \wedge \theta^{Lg^2} \wedge \dots$. On the reduced space (unless for the case of first order polarizations), the existence of a quasi-invariant measure μ is granted, since the wave functions have support on $G/G_{\mathcal{P}}$, which is a homogeneous space (see [7], for instance). This means that the operators X^R , once reduced to \mathcal{H} , are (anti-)hermitian with respect to the scalar product defined by μ , or they can be made (anti-)hermitian by the addition of the Radon-Nikodym derivative in the case where μ is only quasi-invariant. A detailed study of the invariance properties of measures on the space of polarized wave functions is in preparation (see [8]).

The classical theory for the system is easily recovered by defining the Noether invariants as $F_{g^i} \equiv i_{\tilde{X}_{g^i}^R} \Theta$. A Poisson bracket can be introduced (defined by $d\Theta$), in such a way that the Noether invariants generate a Lie algebra isomorphic (if there are no algebraic anomalies, see Sec. 4) to that of \tilde{G} (see [5] for a complete description of the classical theory using the formalism of

GAQ).

2.1 $U(1)$ -central extensions and connection 1-form

To consider the quantization procedure [5, 9], in particular to address the definition of the quantization 1-form Θ and to exhibit the vector fields which are the generators of the desired \tilde{U}_ϕ , we shall be more specific and state our hypothesis more precisely.

Following Bargmann [13], instead of characterizing central extensions by smooth 2-cocycles (*factors*) $\omega : G \times G \rightarrow U(1)$, we shall employ *exponents* $\omega = e^{i\xi}$, i.e. 2-cocycles on G with values in R , $\xi : G \times G \rightarrow R$. In addition, for a simply connected group, any local factor or exponent defined on a neighborhood of the identity can be extended to a global factor or exponent defined on the whole group [13].

In order to select, from the very beginning, the particular projective unitary irreducible representation of G that we want to obtain, we shall consider the central extension \tilde{G} characterized not only by an equivalence class of 2-cocycles $[[\xi]]$ on G (that is, the set of 2-cocycles $\xi : G \times G \rightarrow R$ which differ among them by a coboundary, i.e. a 2-cocycle $\xi_\lambda(g_1, g_2)$ generated by a function $\lambda : G \rightarrow R$, $\xi_\lambda(g_1, g_2) = \lambda(g_1 * g_2) - \lambda(g_1) - \lambda(g_2)$) [13], but by a particular subclass $[\xi]$ of the class $[[\xi]]$. We proceed in this fashion because, as we shall see later, each cohomology class $[[\xi]]$ can be subsequently partitioned into *pseudo-cohomology* classes, which are essentially associated with (classes of) coboundaries generated by functions $\lambda : G \rightarrow R$ with non-zero gradient at the identity e of G .

This implies, in particular, that we can select for $\xi(g_1, g_2)$ a pure coboundary (a 2-cocycle cohomologous to the trivial 2-cocycle $\xi_0(g_1, g_2) \equiv 0$), but belonging to a non-trivial pseudo-cohomology class of coboundaries generated by a function λ with non-zero gradient at the identity. This will be the case, for instance, for (the universal covering groups of) all semisimple (finite-dimensional) groups [14] and the Poincaré group (see Refs. [3] and [15]), all of them with trivial 2^{nd} cohomology group. The advantage of this procedure is that it will allow us to obtain all the unirreps that are included in the regular representation by using the same technique. We should remark that $H^2(G, U(1))$ in the Bargmann cohomology [13] corresponds to the first standard-cohomology group $H^1(G, \mathcal{G}^*)$ (see, for instance, Refs. [16, 3]) of G with values on the co-adjoint module \mathcal{G}^* , which is in turn equivalent to the

first Cartan-Eilenberg cohomology group [14].

Therefore, we shall consider a quantization group \tilde{G} as a central extension of G , connected and simply connected, by $U(1)$ characterized by a 2-cocycle (or exponent) $\xi : G \times G \rightarrow R$, satisfying:

$$\begin{aligned}\xi(g_1, g_2) + \xi(g_1 * g_2, g_3) &= \xi(g_1, g_2 * g_3) + \xi(g_2, g_3) \\ \xi(e, e) &= 0.\end{aligned}\tag{5}$$

Most of the following construction also applies to the case in which \tilde{G} is a non-trivial principal bundle on G , so that a local 2-cocycle must be defined on each fibre-bundle chart. This is the case of Kac-Moody groups (see for instance [17]).

The group law for \tilde{G} can be written (since in all the cases we are considering \tilde{G} is the trivial topological product of G and $U(1)$, the 2-cocycle ξ will be smooth) as:

$$(g', \zeta') * (g, \zeta) = (g' * g, \zeta' \zeta e^{i\xi(g', g)}).\tag{6}$$

Considering a set of local coordinates at the identity $\{g^i, i = 1, \dots, \dim G\}$ in G , the group law can be written as the set of functions $g''^i = g'''^i(g'^j, g^k)$, $j, k = 1, \dots, \dim G$. We introduce the sets of left- and right-invariant vector fields of \tilde{G} associated with the set of coordinates $\{g^i\}$ as those which are written as $\frac{\partial}{\partial g^i}$, $i = 1, \dots, \dim G$ at the identity, that is:

$$\tilde{X}_{g^i}^L(\tilde{g}) = \tilde{L}_g^T \frac{\partial}{\partial g^i}, \quad \tilde{X}_{g^i}^R(\tilde{g}) = \tilde{R}_g^T \frac{\partial}{\partial g^i},\tag{7}$$

where the tilde refers to operations and elements in \tilde{G} . The left- (and right- since the $U(1)$ subgroup is central in \tilde{G}) invariant vector field which at the identity is written as $\frac{\partial}{\partial \phi}$, with $\phi = -i \log \zeta$, is $\tilde{X}_\zeta(\tilde{g}) \equiv 2\text{Re}(i\zeta \frac{\partial}{\partial \zeta}) = \frac{\partial}{\partial \phi}$. We usually keep the sub/superscript ζ instead of ϕ to remind the reader that we are dealing with the compact fibre $U(1)$ and not R . $\tilde{X}_0 \equiv \tilde{X}_\zeta$ is the vertical (or fundamental) vector field associated with the fibre bundle

$$U(1) \rightarrow \tilde{G} \rightarrow G.\tag{8}$$

Analogous considerations can be made for the sets of left- and right-invariant 1-forms associated with the set of local coordinates $\{g^i\}$, i.e. those which at the identity are written as dg^i :

$$\tilde{\theta}^{Lg^i}(\tilde{g}) = \tilde{L}_{g^{-1}}^* dg^i = \theta^{Lg^i}(g), \quad \tilde{\theta}^{Rg^i}(\tilde{g}) = \tilde{R}_{g^{-1}}^* dg^i = \theta^{Rg^i}(g).\tag{9}$$

For simplicity of notation, we shall omit the point in which the vector fields and 1-forms are calculated. Due to the left and right invariance, we have $\theta^{Lg^i}(\tilde{X}_{g^j}^L) = \delta_j^i = \theta^{Rg^i}(\tilde{X}_{g^j}^R)$.

We can also compute the left- and right-invariant 1-forms which are dual to the vertical generator \tilde{X}_0 :

$$\tilde{\theta}^{L\zeta} = \tilde{L}_{\tilde{g}^{-1}}^* d\phi, \quad \tilde{\theta}^{R\zeta} = \tilde{R}_{\tilde{g}^{-1}}^* d\phi. \quad (10)$$

We shall call $\Theta \equiv \tilde{\theta}^{L\zeta} = \frac{d\zeta}{i\zeta} + \frac{\partial \xi(g', g)}{\partial g^i} \big|_{g'=g^{-1}} dg^i$, where $\frac{d\zeta}{i\zeta} = d\phi$, the *quantization 1-form*. It defines a connection on the fibre bundle \tilde{G} and it is uniquely determined by the 2-cocycle $\xi(g_1, g_2)$ (it does not change under changes of local coordinates of G).

Adding to ξ a coboundary ξ_λ , generated by the function λ , results in a new group (note that the new group \tilde{G}' is isomorphic to \tilde{G} , the isomorphism being $(g, \zeta) \mapsto (g, \zeta e^{i\lambda})$) \tilde{G}' and a new quantization 1-form $\Theta' = \Theta + \Theta_\lambda$ with

$$\Theta_\lambda = \lambda_i^0 \theta^{Lg^i} - d\lambda, \quad (11)$$

where $\lambda_i^0 \equiv \frac{\partial \lambda(g)}{\partial g^i} \big|_{g=e}$, that is, the gradient of λ at the identity with respect to a set of local canonical coordinates [13] (see also [18]). Note that Θ is left-invariant under \tilde{G} and Θ' is left-invariant under \tilde{G}' , therefore Θ_λ is neither invariant under \tilde{G} nor \tilde{G}' . However, since λ_i^0 are constants, up to the total differential $d\lambda$, Θ_λ is left invariant under G (and consequently under \tilde{G} and \tilde{G}'). We also have $d\Theta_\lambda = \lambda_i^0 d\theta^{Lg^i}$, so that using the relation

$$d\theta^{Lg^i} = \frac{1}{2} C_{jk}^i \theta^{Lg^j} \wedge \theta^{Lg^k}, \quad (12)$$

where C_{jk}^i , $i, j, k = 1, \dots, \dim G$ are the structure constants of the Lie algebra $\mathcal{G} \equiv T_e G$ in the basis of the left-invariant vector fields associated with the set of local canonical coordinates $\{g^i\}$, we obtain:

$$d\Theta_\lambda = \frac{1}{2} \lambda_i^0 C_{jk}^i \theta^{Lg^j} \wedge \theta^{Lg^k}. \quad (13)$$

Note that λ defines an element $\vec{\lambda}^0$ of the coalgebra \mathcal{G}^* of G characterizing the presymplectic form $d\Theta_\lambda \equiv d\Theta_{\lambda^0}$. It is easy to see then that given $\vec{\lambda}'^0$ and $\vec{\lambda}^0$ on the same orbit of the coadjoint action of G , $\vec{\lambda}'^0 = Ad(g)^* \vec{\lambda}^0$, for some $g \in G$, the corresponding presymplectic forms are related through:

$$d\Theta_{\lambda'^0} \equiv d\Theta_{Ad^*(g)\lambda^0} = Ad^*(g) d\Theta_{\lambda^0} \quad (14)$$

Taking into account that $(G/G_{\lambda^0}, d\Theta_{\lambda^0})$, where G_{λ^0} is the isotropy group in the point $\vec{\lambda}^0$ under the coadjoint action of G , is a symplectic manifold symplectomorphic to the coadjoint orbit through $\vec{\lambda}^0$ with its natural symplectic 2-form ω_{λ^0} , and using proposition 4.4.2 of [17], we can say that ξ_λ is a well-defined 2-cocycle if and only if ω_{λ^0} is of integral class [19].

In summary, we can classify the central extensions of G in equivalence classes, using two kinds of equivalence relations, one subordinated to the other. The first one is the standard one which leads to the 2^{nd} cohomology group $H^2(G, U(1))$, where two 2-cocycles are cohomologous if they differ by a coboundary. According to this we associate with ξ the class $[[\xi]]$, the elements of which differ in a coboundary generated by an arbitrary function on G . With this equivalence class we can associate a series of parameters, given by the corresponding element of $H^2(G, U(1))$, and will be called the *cohomology parameters*. An example of them is the mass parameter which characterizes the central extensions of the Galilei group. However, the previous considerations suggest that each equivalence class $[[\xi]]$ should be further partitioned according to what we shall call *pseudo-cohomology classes*, $[\xi]$, the elements of which differ by a coboundary ξ_α generated by a function α on G having trivial gradient at the identity. Pseudo-cohomology classes are then characterized by coadjoints orbits of \mathcal{G}^* which satisfy the integrality condition (the condition of integrality is associated with the globality of the generating function λ on the group). With these pseudo-cohomology classes we associate a series of parameters called the *pseudo-cohomology parameters*. An example of this is the spin for the Galilei group or the spin and the mass for the Poincaré group. Note that these parameters are associated with (integral) coadjoint orbits of the correspondig groups.

The idea of subclasses inside $H^2(G, U(1))$ was firstly introduced by Saletan [11] who noted that under an Inönü-Wigner contraction some coboundaries ξ_λ of G become non-trivial cocycles ξ_c of the contracted group G_c since the generating function λ is badly behaved in the contraction limit while ξ_λ itself has a well defined limit ξ_c . The simplest physical example is that of the Poincaré group whose pseudo-cohomology group goes to the cohomology group of the Galilei group [12]. For semisimple groups, pseudo-cohomology is also related to the Čech cohomology of the generalized Hopf fibration by the Cartan subgroup H , $G \rightarrow G/H$, [10].

Some comments on pseudo-cohomology classes are in order. Firstly, in

each pseudo-cohomology class one can always find representatives that are linear in the coordinates $\{g^i\}$ in a neighbourhood of the identity, that is, if ξ_λ is a pseudo-cocycle with generating function $\lambda(g)$ having gradient $\vec{\lambda}^0$ at the identity, then ξ_λ is pseudo-cohomologous to the pseudo-cocycle generated by $\lambda_i^0 g^i$, since $\lambda(g) - \lambda_i^0 g^i$ has zero gradient at the identity of the group.

Secondly, pseudo-cocycles ξ_λ associated with those points $\vec{\lambda}^0 \in \mathcal{G}^*$ that are invariant under the coadjoint action of G (i.e. they constitute zero-dimensional coadjoint orbits) are either zero or belong to the trivial pseudo-cohomology class, since for them $d\Theta_\lambda = 0$. Therefore, zero-dimensional coadjoint orbits do not lead to central (pseudo-)extensions. This is reasonable since zero-dimensional coadjoint orbits are associated with one-dimensional representations, which are abelian. However, pseudo-cocycles associated with zero-dimensional coadjoint orbits, in certain cases, play an important role in the process of unitarizing representations (see [8]) for which an invariant measure on the representation space does not exist (and only quasi-invariant ones can be found). The unitarization process involves central pseudo-extensions by the abelian multiplicative group R^+ leading to the Radon-Nikodym derivative.

Finally, the quantum operators \tilde{X}_i^R obtained from the quantization of a pseudo-extended group should be redefined with the addition of the linear terms $\lambda_i^0 \tilde{X}_0$, i.e. $\tilde{X}_i^{R'} \equiv \tilde{X}_i^R + \lambda_i^0 \tilde{X}_0$, in order to obtain the original commutations relations (see the example of $SU(2)$ in Sec. 3.2).

In the general case, including infinite-dimensional semi-simple Lie groups, for which the Whitehead lemma does not apply, the group law for \tilde{G} will contain cocycles as well as pseudo-cocycles (see [20, 10, 6, 21]). The simplest physical example of a quantum symmetry including such an extension is that of the free non-relativistic particle with spin; the Galilei group must be extended by a true cocycle to describe the canonical commutation relations between q's and p's as well as by a pseudo-cocycle associated with the Cartan subgroup of $SU(2)$, to account for the spin (we shall consider the universal covering of the Galilei group, so that we obtain $SU(2)$ as the rotation group and therefore half-integer values are allowed for the spin) degree of freedom [15].

Let us therefore consider a Lie group \tilde{G} which is a $U(1)$ -principal bundle with the bundle projection $\pi : \tilde{G} \rightarrow G$ being a group homomorphism. We denote by Θ the connection 1-form constructed as explained earlier. It

satisfies $i_{\tilde{X}_0}\Theta = 1$, $L_{\tilde{X}_0}\Theta = 0$, with \tilde{X}_0 being the infinitesimal generator of $U(1)$, or the fundamental vector field of the principal bundle, which is in the centre of the Lie algebra $\tilde{\mathcal{G}} \equiv T_e\tilde{G}$. Since Θ is left-invariant it will be preserved ($L_{\tilde{X}_R}\Theta = 0$) by all right-invariant vector fields (generating finite left translations) on \tilde{G} . These vector fields are candidates to be infinitesimal generators of unitary transformations. To define the space of functions on which they should act, we proceed as follows. Choose a representation of the structure group $U(1)$, which will be the natural representation on the complex numbers, and build the space of complex functions on \tilde{G} that satisfy the $U(1)$ -equivariance condition

$$L_{\tilde{X}_0}\psi = i\psi. \quad (15)$$

where \tilde{X}_0 is the fundamental vector field on the principal bundle $\tilde{G} \rightarrow G$. This space is isomorphic to the linear space of sections of the bundle $E \rightarrow \tilde{G}/U(1)$, with fibre $F \equiv C$, associated with $\tilde{G} \rightarrow G$ through the natural representation of $U(1)$ on the complex numbers C (see, for instance, [22]). To get an irreducible action of the right-invariant vector fields we have to select appropriate subspaces, and this will be achieved by polarization conditions.

2.2 Additional structures associated with the connection 1-form

The 2-form $\tilde{\Sigma} \equiv d\Theta$ is left invariant under \tilde{G} , and is projectable to a left invariant 2-form Σ of G . This, evaluated at the identity, defines a 2-cocycle on the Lie algebra \mathcal{G} .

On vector fields $\mathcal{X}(G)$ we can define a “generalized Lagrange bracket” by setting, for any pair of vector fields $X, Y \in \mathcal{X}(G)$,

$$(X, Y)_\Sigma = \Sigma(X, Y) \in \mathcal{F}(G). \quad (16)$$

In particular, when we consider left invariants vector fields $X^L, Y^L \in \mathcal{X}^L(G)$, we get a real valued bracket:

$$(X^L, Y^L)_\Sigma = \Sigma(X^L, Y^L) \in R. \quad (17)$$

By evaluating $\tilde{\Sigma}$ at the identity of the group, i.e. on $T_e\tilde{G} = \tilde{\mathcal{G}}$, we can bring it to normal form which would be the analog of a Darboux frame in

the space of left-invariant 1-forms. We can write

$$\tilde{\Sigma} = \sum_{a=1}^k \theta^{La} \wedge \theta^{La+k}, \quad (18)$$

where $\theta^{La}, \theta^{La+k}$, $a = 1, \dots, k$ are left-invariant 1-forms. We can define a (1,1)-tensor field J , a partial (almost) complex structure, by setting:

$$\begin{aligned} J\theta^{La} &= \theta^{La+k} \\ J\theta^{La+k} &= -\theta^{La} \\ J\theta^{Ll} &= 0, \end{aligned} \quad (19)$$

where θ^{Ll} are the remaining elements of a basis of left-invariant 1-forms not appearing in $\tilde{\Sigma}$ (that is, dual to vector fields in $\text{Ker}\tilde{\Sigma}$). We also have a “partial metric tensor” ρ by setting $\rho(\theta^{La}, \theta^{La'}) = \delta_{aa'}$, $\rho(\theta^{La}, \theta^{Ll}) = 0$, $\rho(\theta^{Ll}, \theta^{Ll'}) = 0$.

Our considerations will be always restricted to finite-dimensional Lie groups or infinite-dimensional ones possessing a countable basis of generators for which, for arbitrary fixed \tilde{X}^L , $\tilde{\Sigma}(\tilde{X}^L, \tilde{Y}^L) = 0$ except for a finite number of vector fields \tilde{Y}^L (finitely non-zero cocycle), and therefore this partial (almost) complex structure J can always be introduced.

It is possible to associate with Θ an horizontal projector, a (1,1)-tensor field. We first define the vertical projector $V_{\Theta}(X) = \Theta(X)X_0$, and then $H_{\Theta} = I - V_{\Theta}$.

The characteristic module of Θ is defined to be the intersection of $\text{Ker}\Theta$ and $\text{Ker}d\Theta = \text{Ker}\tilde{\Sigma}$. By restricting to $\mathcal{X}^L(\tilde{G})$ we get the characteristic subalgebra \mathcal{G}_C . Elements in $\text{Ker}\Theta \cap \text{Ker}d\Theta$ are easily shown to be a Lie algebra. In fact, it follows from the identity $d\Theta(X, Y) = L_X\Theta(Y) - L_Y\Theta(X) - \Theta([X, Y])$.

It turns out that the quotient of \tilde{G} by the integrable distribution generated by \mathcal{G}_C , $P \equiv \tilde{G}/\mathcal{G}_C$, is a quantum bundle in the sense of Geometric Quantization, with connection the projection of Θ to P (see [5]). Therefore, $d\Theta$ projected onto $P/U(1)$ is a symplectic 2-form, establishing the connection with the Coadjoint Orbits Method, the different coadjoint orbits being obtained by suitable choice of the (pseudo)extension parameters. To be more specific, if Ω_{μ} is the orbit through an arbitrary point $\mu \in \mathcal{G}^*$, we can construct the left-invariant 1-form $\theta_{\mu}^L \in \mathcal{X}^L(G)^*$ and $\omega_{\mu} \equiv d\theta_{\mu}^L$. Then, the quotient of

G by the characteristic distribution of ω_μ , $G/\text{Ker}\{\omega_\mu\}$ is differentiably symplectomorphic to Ω_μ :

$$(\Omega_\mu, \mu.[,]) \approx (G/\text{Ker}\{\omega_\mu\}, \omega_\mu), \quad (20)$$

where $\mu.[,]$ is the standard symplectic form on the coadjoint orbits. Note that had we taken \tilde{G} instead of G , $\mu.[,]$ would have been replaced by $\tilde{\mu}.[,] = \mu.[,] + \tilde{\Sigma}$. By considering functions on this orbit, we have the possibility of considering them as functions on G (by taking the corresponding pull-back). Now, to these functions we can apply the left and right action of G or, even, the operators in the enveloping (left and right) algebra, and these operators are not available in \mathcal{G}^* .

However, we are not going to consider such a quotient explicitly. Rather, the inclusion of the characteristic subgroup in the pre-contact manifold \tilde{G} represents a non-trivial improvement and generalization of Geometric Quantization in the sense that equations of motion can be naturally included into the quantization scheme, and also because we are not forced, this way, to consider the classical equations of motion, the solutions of which might be lacking.

2.3 Polarizations

As commented at the beginning of Sec. 2, to reduce the representation constituted by the left action of the group \tilde{G} on wave functions satisfying (15), we need to select appropriate invariant subspaces. This is achieved by means of the polarization conditions, in terms of suitable subalgebras of left-invariant vector fields. Thus, a **first-order polarization** or just **polarization** \mathcal{P} is defined as a maximal horizontal left subalgebra. The horizontality condition means that the polarization is in $\text{Ker}\Theta$. Again, by using the identity $d\Theta(X, Y) = L_X\Theta(Y) - L_Y\Theta(X) - \Theta([X, Y])$ we find that the generalized Lagrange bracket of any two elements of \mathcal{P} vanishes. Therefore we find that a polarization is an isotropic maximal subalgebra. We notice that maximality is with respect to the Lie commutator (subalgebra) not with respect to isotropy (Lagrange bracket).

A polarization may have non-trivial intersection with the characteristic subalgebra. We say that a polarization is **full (or regular)** if it contains the whole characteristic subalgebra. We also say that a polarization \mathcal{P} is **sym-**

plectic if $\tilde{\Sigma}$ on $\mathcal{P} \oplus J\mathcal{P}$ is of maximal rank. Full and symplectic polarizations correspond to *admissible* subalgebras subordinated to $\Theta|_e \in \tilde{\mathcal{G}}^*$ [19].

It should be stressed that the notion of polarization and characteristic subalgebras here given in terms of Θ is really a consequence of the fibre bundle structure of the group law of \tilde{G} and, therefore, can be translated into finite (versus infinitesimal) form defining the corresponding subgroups (see [9]).

From the geometric point of view, a polarization defines a foliation via the Frobenius theorem. It is possible to select subspaces of equivariant complex valued functions on \tilde{G} , by requiring them to be constant along integral leaves of the foliation associated with the polarization. Whether this subspace is going to carry an irreducible representation for the right-invariant vector fields is to be checked. When the polarization is full and symplectic we get leaves which are maximally isotropic submanifolds for $d\Theta$. The selected subspaces of equivariant complex valued functions on \tilde{G} , which we may call wave functions, will be characterized by $L_{\tilde{X}_0}\Psi = i\Psi$, $L_{X^L}\Psi = 0$, $\forall X^L \in \mathcal{P}$.

Remark: We can generalize the notion of polarization by simply relaxing the condition of horizontality, and defining a **non-horizontal polarization** as a maximal left subalgebra not containing the vertical generator \tilde{X}_0 . Although this kind of polarizations are not horizontal with respect to the quantization 1-form Θ , it is always possible to find a new Θ' for which a given non-horizontal polarization becomes horizontal, and Θ' is of the form:

$$\Theta' = \Theta + \alpha_i \theta^{Lg^i}, \quad (21)$$

which implies that, up to a total differential, Θ' is obtained adding a coboundary (pseudococycle) to the original 2-cocycle. Therefore, the description in terms of pseudo-extensions and that of non-horizontal polarizations are equivalent.

3 Simple examples

To see how this construction works, let us consider in detail two paradigmatic examples: the abelian group R^k and the semisimple one $SU(2)$. These suffice to illustrate GAQ in its easiest (first-order) form. Anomalous cases will be encountered in the next sections.

3.1 The abelian group R^k

Despite of the fact that R^k is the simplest locally compact group, it deserves a detailed study since its non-trivial (projective) representations, the group being non-compact, are infinite-dimensional, leading to the well-known Heisenberg-Weyl commutation relations, the base for non-relativistic Quantum Mechanics.

We shall parameterize R^k by (global in this case) canonical coordinates $\vec{x} = (x^1, \dots, x^k)$. Since the group is abelian, the coadjoint action is trivial and its coadjoint orbits are points (zero-dimensional). Therefore, there will not be pseudo-cohomology classes and only the cohomology group is relevant. This means that given any 2-cocycle ξ defining a central extension \tilde{R}^k of R^k , $\Theta - \frac{d\xi}{i\zeta}$, where Θ is the quantization 1-form, is not left invariant under the action of R^k , that is, under translations (not even up to a total differential). Since $d\Theta$ is always left invariant, we see that it is an exact 2-form but it is not invariantly exact, this fact being a consequence of the non-trivial group cohomology of R^k . Since the group is abelian, $d\Theta$ takes the same value at all points, and can be written as:

$$d\Theta = a_{ij}dx^i \wedge dx^j, \quad (22)$$

where a_{ij} is an antisymmetric $k \times k$ matrix. This allows us to write any 2-cocycle in the form:

$$\xi(\vec{x}_1, \vec{x}_2) = a_{ij}x_1^i x_2^j, \quad (23)$$

up to a coboundary which, due to the trivial pseudo-cohomology of R^k , will always contribute to Θ with an irrelevant total differential. Therefore, ξ is an antisymmetric bilinear function on R^k , and with an appropriate change of coordinates in R^k can be taken to normal form, in which the matrix a_{ij} is written as:

$$\frac{1}{2} \left(\begin{array}{ccc|ccc|c} 0_n & & & D_n & & & \\ \hline - & - & - & - & - & - & \vec{0}_{2n} \\ \hline -D_n & & & 0_n & & & \\ \hline - & - & - & - & - & - & \\ & & & {}^t\vec{0}_{2n} & & & \\ & & & & & & 0_r \end{array} \right), \quad (24)$$

where 0_p is the $p \times p$ zero matrix, $\vec{0}_{2n}$ is the zero $2n$ -dimensional column

vector, and D_n is a $n \times n$ real matrix of the form:

$$\begin{pmatrix} \nu_1 & 0 & . & . & . & 0 \\ 0 & \nu_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & \nu_n \end{pmatrix}, \quad (25)$$

with $k = 2n + r$. The parameters ν_1, \dots, ν_n characterize the extension \tilde{R}^k , and thus they are the cohomology parameters. In all physical situations, the subspace R^{2n} of R^k is associated with the phase-space of a physical system, which possesses other symmetries than those of R^k , like, for instance, rotations in each one of the subspaces R^n of R^{2n} . By requiring isotropy will fix these parameters to coincide, $\nu_i = \nu, \forall i = 1, \dots, n$. For this case, the 2-cocycle can be written as:

$$\xi(\vec{q}_1, \vec{p}_1, \vec{a}_1; \vec{q}_2, \vec{p}_2, \vec{a}_2) = \frac{1}{2} \nu (\vec{q}_2 \cdot \vec{p}_1 - \vec{q}_1 \cdot \vec{p}_2), \quad (26)$$

where \vec{q}_i are n -dimensional vectors corresponding to the first n coordinates (in the new basis), \vec{p}_i correspond to the following n coordinates, and \vec{a}_i to the remaining r coordinates. Note that ξ does not depend on the \vec{a}_i , so the group \tilde{R}^k can be written as $\text{H-W}_n \times R^r$, where H-W_n is the well-known Heisenberg-Weyl group. The group law for \tilde{R}^k can be written, in these new coordinates, as:

$$\begin{aligned} \vec{q}'' &= \vec{q}' + \vec{q} \\ \vec{p}'' &= \vec{p}' + \vec{p} \\ \vec{a}'' &= \vec{a}' + \vec{a} \\ \zeta'' &= \zeta' \zeta e^{\frac{i}{2} \nu (\vec{p}' \cdot \vec{q} - \vec{q}' \cdot \vec{p})}. \end{aligned} \quad (27)$$

From the group law we see that if \vec{q} is interpreted as coordinates, and \vec{p} as momenta, then $\nu = \hbar^{-1}$. Therefore, the cohomology parameter for the (isotropic) Heisenberg-Weyl group can be identified with \hbar . The variables \vec{a} do not play any role, and can be factorized, as we will see later.

Left and right invariant (under \tilde{R}^k) vector fields are:

$$\begin{aligned}\tilde{X}_{\vec{q}}^L &= \frac{\partial}{\partial \vec{q}} + \frac{\vec{p}}{2\hbar} \tilde{X}_0 & \tilde{X}_{\vec{q}}^R &= \frac{\partial}{\partial \vec{q}} - \frac{\vec{p}}{2\hbar} \tilde{X}_0 \\ \tilde{X}_{\vec{p}}^L &= \frac{\partial}{\partial \vec{p}} - \frac{\vec{q}}{2\hbar} \tilde{X}_0 & \tilde{X}_{\vec{p}}^R &= \frac{\partial}{\partial \vec{p}} + \frac{\vec{q}}{2\hbar} \tilde{X}_0 \\ \tilde{X}_{\vec{a}}^L &= \frac{\partial}{\partial \vec{a}} & \tilde{X}_{\vec{a}}^R &= \frac{\partial}{\partial \vec{a}},\end{aligned}\tag{28}$$

and $\tilde{X}_0 = \frac{\partial}{\partial \phi}$ is the vertical (left and right invariant) vector field. The commutation relations for these vector fields are:

$$[\tilde{X}_{q^i}^L, \tilde{X}_{p^j}^L] = -\frac{1}{\hbar} \tilde{X}_0, \tag{29}$$

the rest of them being zero. In this way we reproduce the standard Weyl commutation relations. Left and right invariant 1-forms for R^k are simply $d\vec{x}$, for $\vec{x} = \vec{q}, \vec{p}$ and \vec{a} . The quantization 1-form Θ , which for convenience we redefine with a factor \hbar , is:

$$\Theta = \hbar \frac{d\zeta}{i\zeta} + \frac{1}{2}(\vec{q} \cdot d\vec{p} - \vec{p} \cdot d\vec{q}). \tag{30}$$

Note that $d\Theta = d\vec{q} \wedge d\vec{p}$ is a pre-symplectic form on R^k , with kernel the subspace R^r spanned by the vectors \vec{a} . On the quotient $R^k/R^r = R^{2n}$, it is a true symplectic form. In fact, a partial complex structure J can be introduced, of the form $J = dp^i \otimes \tilde{X}_{q^i}^L - dq^i \otimes \tilde{X}_{p^i}^L$. J turns to be a complex structure on the reduced space R^k/R^r .

The characteristic subalgebra, i.e. $\text{Ker}\Theta \cap \text{Ker}d\Theta$, is therefore $\mathcal{G}_\Theta = \langle \tilde{X}_{\vec{a}}^L \rangle$, and the possible horizontal polarizations we can find are of the form:

$$\mathcal{P} = \langle \tilde{X}_{\vec{a}}^L, \alpha_i \tilde{X}_{q^i}^L + \beta_i \tilde{X}_{p_i}^L, i = 1, \dots, n \rangle, \tag{31}$$

with restrictions on the real coefficients α_i, β_i such that it is maximal, and horizontal with respect to Θ . These restrictions imply that the polarizations are full and symplectic. There are two selected polarizations, $\mathcal{P}_p = \langle \tilde{X}_{\vec{a}}^L, \tilde{X}_{\vec{q}}^L \rangle$ and $\mathcal{P}_q = \langle \tilde{X}_{\vec{a}}^L, \tilde{X}_{\vec{p}}^L \rangle$ leading to the representations in momentum and configuration space, respectively. It should be stressed that all this polarizations lead to equivalent representations of R^k , and in particular the unitary operator relating the representations obtained with \mathcal{P}_p and \mathcal{P}_q is the Fourier transform. This is an outer isomorphism of H-W_n , but it is inner in the Weyl-Symplectic group $WSp(2n, R)$ [23].

Taking advantage of the natural complex structure of $R^{2n} \approx C^n$ (the one induced by J), we can allow for a complex polarization of the form:

$$\mathcal{P}_c = \langle \tilde{X}_a^L, \tilde{X}_{\vec{q}}^L + i\mu\tilde{X}_{\vec{p}}^L \rangle, \quad (32)$$

where μ is a constant with the appropriate dimensions (from the physical point of view, it will be a mass times a frequency, which makes this polarization appropriate for the description of the Harmonic Oscillator, see Sec. 3). This polarization leads to a representation in terms of holomorphic (or anti-holomorphic) functions on C^n . It is unitarily equivalent to the other representations, the unitary transformation which relates it with the representation in configuration space being the Bargmann transform. This is also an outer automorphism of $H\text{-}W_n$, but it is inner in a certain subsemigroup of $Sp(2n, C)$ [23].

Let us compute, for instance, the representation obtained with the polarization \mathcal{P}_q . The equations $\tilde{X}_a^L\Psi = 0$ leads to wave functions not depending on the \vec{a} variables (they trivially factorize), and the equations $\tilde{X}_p^L\Psi = 0$ lead to (together with the equivariance condition $\tilde{X}_0\Psi = i\Psi$):

$$\Psi = \zeta e^{\frac{i}{2\hbar}\vec{q}\cdot\vec{p}}\Phi(\vec{q}), \quad (33)$$

where $\Phi(\vec{q})$ is an arbitrary function of \vec{q} (apart from normalizability considerations). If we compute the action of the right-invariant vector fields on these wave functions, we obtain:

$$\tilde{X}_{\vec{q}}^R\Psi = \zeta e^{\frac{i}{2\hbar}\vec{q}\cdot\vec{p}}\left(\frac{\partial}{\partial\vec{q}}\Phi(\vec{q})\right) \quad (34)$$

$$\tilde{X}_{\vec{p}}^R\Psi = \zeta e^{\frac{i}{2\hbar}\vec{q}\cdot\vec{p}}\left(-\frac{i}{\hbar}\vec{q}\Phi(\vec{q})\right). \quad (35)$$

This representation is unitarily equivalent to the Schrödinger representation (for each value of the cohomology parameter \hbar), with respect to the measure $\mu = dq^1 \wedge \cdots \wedge dq^n$, which in this case is invariant under the group \tilde{G} . This measure can be obtained contracting the left Haar measure Ω^L with respect to all the vector fields in the polarization.

3.2 The semisimple group $SU(2)$

Let us consider now an example, which in a certain sense is on the other extreme to that of the abelian group R^k . It is the semisimple group $SU(2)$, which has trivial cohomology group $H^2(SU(2), U(1)) = \{0\}$. For this reason all 2-cocycles on $SU(2)$ are coboundaries, and they will be classified according to pseudocohomology classes only.

Making use of the realization of $SU(2)$ as 2×2 complex matrices of the form:

$$\begin{pmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{pmatrix}, \quad (36)$$

with $|z_1|^2 + |z_2|^2 = 1$, with matrix multiplication as group law, we introduce stereographic projection coordinates $\{\eta \equiv \frac{z_2}{|z_1|}, c \equiv \frac{z_2}{z_1}, c^* = (c)^*\}$, which are defined for $z_1 \neq 0$. For $z_1 = 0$ another chart is needed, but we shall forget about it and make use of the well-know geometrical properties of the sphere to obtain the relevant results.

The group law in these coordinates is written as:

$$\begin{aligned} \eta'' &= \frac{\eta'\eta - \eta'^* \eta c'^* c}{\sqrt{(1 - \eta'^2 c' c^*)(1 - \eta'^*{}^2 c'^* c)}} \\ c'' &= \frac{c' \eta'^2 + c}{\eta'^2 - c'^* c} \\ c^{*''} &= (c'')^*. \end{aligned} \quad (37)$$

This set of coordinates has been chosen to make explicit the fibre bundle structure of $SU(2)$ over the sphere, $U(1) \rightarrow SU(2) \rightarrow S^2$, where $\eta \in U(1)$ is the parameter of the fibre and the bundle projection is $\pi(\eta, c, c^*) = (c, c^*)$, with c, c^* the stereographic projection coordinates of the sphere on the complex plane. The fibre bundle structure is with respect to the right action of $SU(2)$, $(\eta', c', c'^*) * (\eta, 0, 0) = (\eta'\eta, c', c'^*)$.

As we mentioned before, due to the trivial cohomology of $SU(2)$, the only 2-cocycles on it are coboundaries. They are classified according to pseudo-cohomology classes, which are in one-to-one correspondence with integral co-adjoint orbits of $SU(2)$.

If we write $\eta = e^{i\varphi}$, then we can choose as representatives for the pseudo-cohomology classes $\xi_{2j}(g', g) = 2j(\varphi'' - \varphi' - \varphi)$. Using this pseudo-cocycle,

we can introduce the following group law in the direct product $SU(2) \times U(1)$:

$$g'' = g' * g, \quad \zeta'' = \zeta' \zeta e^{i2j(\varphi'' - \varphi' - \varphi)}. \quad (38)$$

It is interesting to note that this is well-defined and therefore a group law, or, in other words, ξ_{2j} satisfies the 2-cocycle properties only if $2j \in Z$ (see (5)), from which we obtain the correct quantization condition of j from the beginning, using only pseudo-cohomology considerations.

Left-invariant vector fields are given by:

$$\begin{aligned} \tilde{X}_\eta^L &= \frac{\partial}{\partial \varphi} \\ \tilde{X}_c^L &= \eta^{-2} \left[(1 + |c|^2) \frac{\partial}{\partial c} + \frac{i}{2} c^* \left(\frac{\partial}{\partial \varphi} + 2j \tilde{X}_0 \right) \right] \\ \tilde{X}_{c^*}^L &= \eta^2 \left[(1 + |c|^2) \frac{\partial}{\partial c^*} - \frac{i}{2} c \left(\frac{\partial}{\partial \varphi} + 2j \tilde{X}_0 \right) \right], \end{aligned} \quad (39)$$

and right-invariant ones by:

$$\begin{aligned} \tilde{X}_\eta^R &= \frac{\partial}{\partial \varphi} - 2ic \frac{\partial}{\partial c} + 2ic^* \frac{\partial}{\partial c^*} \\ \tilde{X}_c^R &= \frac{\partial}{\partial c} + c^{*2} \frac{\partial}{\partial c^*} - \frac{i}{2} c^* \left(\frac{\partial}{\partial \varphi} + 2j \tilde{X}_0 \right) \\ \tilde{X}_{c^*}^R &= \frac{\partial}{\partial c^*} + c^2 \frac{\partial}{\partial c} + \frac{i}{2} c \left(\frac{\partial}{\partial \varphi} + 2j \tilde{X}_0 \right). \end{aligned} \quad (40)$$

The commutation relations that these vector fields satisfy are:

$$\begin{aligned} [\tilde{X}_\eta^L, \tilde{X}_c^L] &= -2i \tilde{X}_c^L \\ [\tilde{X}_\eta^L, \tilde{X}_{c^*}^L] &= 2i \tilde{X}_{c^*}^L \\ [\tilde{X}_c^L, \tilde{X}_{c^*}^L] &= -i(\tilde{X}_\eta^L + 2j \tilde{X}_0). \end{aligned} \quad (41)$$

The left-invariant 1-forms, which are dual to the $\tilde{X}_{g^i}^L$'s, are:

$$\begin{aligned} \theta^{L\eta} &= \frac{d\eta}{i\eta} - \frac{1}{2} \frac{ic^*}{1 + |c|^2} dc + \frac{1}{2} \frac{ic}{1 + |c|^2} dc^* \\ \theta^{Lc} &= \frac{\eta^2}{1 + |c|^2} dc \\ \theta^{Lc^*} &= \frac{\eta^{-2}}{1 + |c|^2} dc^*, \end{aligned} \quad (42)$$

and the quantization 1-form is:

$$\Theta = \frac{d\zeta}{i\zeta} - ij \frac{cdc^* - c^*dc}{1 + |c|^2}. \quad (43)$$

The partial complex structure J is given by $J = \theta^{Lc^*} \otimes \tilde{X}_c^L - \theta^{Lc} \otimes \tilde{X}_{c^*}^L$. Note that $d\Theta = 2ij \frac{dc \wedge dc^*}{(1+|c|^2)^2}$ is a pre-symplectic form on $SU(2)$ which is projectable to S^2 defining the standard symplectic structure on the sphere. The characteristic subalgebra is $\mathcal{G}_C = \langle \tilde{X}_\eta^L \rangle$, if $j \neq 0$ and the whole $SU(2)$ algebra if $j = 0$. We shall not consider this last case, since it corresponds to the trivial representation of $SU(2)$ (with spin zero), the wave functions being constants on the group and the operators being zero. Therefore, let us suppose $j \neq 0$. Then there are two possible polarizations, both full and symplectic:

$$\begin{aligned} \mathcal{P}_c &= \langle \tilde{X}_\eta^L, \tilde{X}_{c^*}^L \rangle \\ \mathcal{P}_{c^*} &= \langle \tilde{X}_\eta^L, \tilde{X}_c^L \rangle. \end{aligned} \quad (44)$$

These two polarizations lead to equivalent representations, since there is an inner automorphism which takes one into the other. The first leads to a holomorphic representation, while the second one leads to an anti-holomorphic one, and the isomorphism is given by $(\eta, c, c^*) \rightarrow (\eta^*, c^*, c)$.

Thus, we shall solve only the holomorphic polarization \mathcal{P}_c , with solutions

$$\Psi = \zeta(1 + |c|^2)^{-j} \Phi(c). \quad (45)$$

If we had not obtained the integrality condition $2j \in \mathbb{Z}$ by pseudo-cohomology considerations (if we had used a non-horizontal polarization in the direct product $SU(2) \times U(1)$ with quantization 1-form $\Theta = \frac{d\zeta}{i\zeta}$), we would obtain it by geometrical considerations, i.e through the chart compatibility on the sphere. Making use of it, we also obtain that the function $\Phi(c)$ is of the form:

$$\Phi(c) = \sum_{l=0}^{2j} A_l c^l, \quad (46)$$

with arbitrary coefficients A_l , from which we recover the result that the dimension of the representation labelled by j is $2j+1$. Note that the chart compatibility is what prevents the appearance of non-trivial null states in going

from the Lie algebra to the Lie group level, and this is also true for infinite-dimensional representations of non-compact Lie groups, such as $SL(2, R)$ (see [24], for instance), and even for infinite-dimensional Lie groups such as the Virasoro group [20].

The representation is given in terms of the action of the right-invariant vector fields (which preserve the Hilbert space of polarized wave functions):

$$\begin{aligned}\tilde{X}_\eta^R \Psi &= \zeta(1 + |c|^2)^{-j} (-2ic\dot{\Phi}) \\ \tilde{X}_c^R \Psi &= \zeta(1 + |c|^2)^{-j} \dot{\Phi} \\ \tilde{X}_{c^*}^R \Psi &= \zeta(1 + |c|^2)^{-j} (c^2\dot{\Phi} - 2jc\Phi) ,\end{aligned}\tag{47}$$

where $\dot{\Phi} = d\Phi(c)/dc$. \tilde{X}_η^R is a diagonal operator (generator of the Cartan subalgebra), \tilde{X}_c^R can be interpreted as an annihilation operator and $\tilde{X}_{c^*}^R$ as a creation operator. Since the representation is finite-dimensional, there are maximal and minimal weight states, given by $\Phi = c^j$ and $\Phi = 1$, respectively. Note that we reproduce the standard commutation relations for $SU(2)$ in terms of creation and annihilation operators with the definitions:

$$\begin{aligned}\hat{J}_0 &\equiv \frac{i}{2}(\tilde{X}_\eta^R + 2j\tilde{X}_0) \\ \hat{J}_+ &\equiv \frac{i}{\sqrt{2}}\tilde{X}_{c^*}^R \\ \hat{J}_- &\equiv \frac{i}{\sqrt{2}}\tilde{X}_c^R\end{aligned}\tag{48}$$

This implies, in particular, that the generator \tilde{X}_η^R should be redefined by $\tilde{X}_\eta^{R'} = \tilde{X}_\eta^R + 2j\tilde{X}_0$, in accordance with the general theory, since in this case $\lambda_\eta^0 = 2j$, $\lambda_c^0 = \lambda_{c^*}^0 = 0$.

There is an invariant measure of the form $\mu = \theta^{Lc} \wedge \theta^{Lc^*} = \frac{dc \wedge dc^*}{(1+|c|^2)^2}$, which is obtained by contracting the left Haar measure Ω^L with \tilde{X}_η^L . With respect to this measure \hat{J}_0 is hermitian, and \hat{J}_+ and \hat{J}_- are the adjoint of each other.

4 Algebraic Anomalies

In Sec.2, we introduced the concept of full and symplectic polarization subalgebra intended to reduce the representation obtained through the right-

invariant vector fields acting on equivariant functions on the group. It contains “half” of the symplectic vector fields as well as the entire characteristic subalgebra. If the full reduction is achieved, the whole set of physical operators can be rewritten in terms of the basic ones, i.e. those which are the right version of the left-invariant generators in $J\mathcal{P} \oplus J^2\mathcal{P}$. For instance, the energy operator for the free particle can be written as $\frac{\hat{p}^2}{2m}$, the angular momentum in 3+1 dimensions is the vector product $\hat{\mathbf{x}} \times \hat{\mathbf{p}}$, or the energy for the harmonic oscillator is $\hat{c}^\dagger \hat{c}$ (note that, since we are using first-order polarizations, all this operators are really written as first-order differential operators, and, for instance, the energy operator in momentum space is written as $\hat{E}\Psi = \frac{p^2}{2m}\Psi$, which is a zeroth-order differential operator, indeed).

However, the existence of a full and symplectic polarization is guaranteed only for semisimple and solvable groups [19]. We define an **anomalous** group [6] (see also [25, 26]) as a group G which, for some central extension \tilde{G} characterized by certain values of the (pseudo-)cohomology parameters, does not admit any polarization which is full and symplectic. These values of the (pseudo-)cohomology parameters are called the **classical** values of the anomaly, because they are associated with some coadjoint orbits of the group \tilde{G} (generally exceptional orbits, of lower dimension), that is, with the classical phase space of some physical system (see the discussion in Sec. 2 on the relation between (pseudo-)cohomology parameters and coadjoint orbits of the group \tilde{G}).

Anomalous groups feature another set of values of the (pseudo-)cohomology parameters, called the **quantum** values of the anomaly, for which the carrier space associated with a full and symplectic polarization contains an invariant subspace. For the classical values of the anomaly, the classical solution manifold undergoes a reduction in dimension thus increasing the number of (non-linear) relationships among Noether invariants (invariant relations which characterize the lower dimensional exceptional orbits. These can be defined as a set of equations of the form $f_i(F_{g^i}) = 0, i = 1, \dots, k$, where the functions $f_i : \tilde{\mathcal{G}}^* \rightarrow R$ are in involution and satisfy $\{f_i, F_{g^i}\} = \gamma_{ij}^k f_k$ with γ_{ij}^k functions on $\tilde{\mathcal{G}}$. See [27] for a discussion on invariant relations in the context of Rational Mechanics), whereas for the quantum values the number of basic operators decreases on the invariant (reduced) subspace due to the appearance of (higher-order) relations among the quantum operators (“quantum invariant relations”, which can be defined as a set of equations of the

form $A_i \Psi = 0, i = 1, \dots, k$, where $A_i \in \mathcal{U}\tilde{\mathcal{G}}^L$ close an algebra and satisfy $[A_i, \tilde{X}_{g^j}^L] = B_{ij}^k A_k$, with $B_{ij}^k \in \mathcal{U}\tilde{\mathcal{G}}^L$. The anomaly lies in the fact that the classical and quantum values of the anomalies do not coincide, but there is a "shift" between them or even there is no relation at all among them. The reason is that, when passing from the classical invariant relations to the quantum ones, problems of normal ordering can appear which "deform" the classical Poisson algebra between the functions defining the classical invariant relations.

We must remark that the anomalies we are dealing with in this paper are of *algebraic* character in the sense that they appear at the Lie algebra level, and must be distinguished from the *topologic anomalies* which are associated with the non-trivial homotopy of the (reduced) phase space [28].

The non-existence of a full and/or symplectic polarization for certain values of the (pseudo-)cohomology parameters (the classical values of the anomaly) is traced back to the presence in the characteristic subalgebra of some elements the adjoint action of which are not diagonalizable in the complementary subspace of \mathcal{G}_C in $\tilde{\mathcal{G}}$. In other words, no maximal isotropic subspace for the symplectic 2-form on the coadjoint orbit is a Lie subalgebra. The anomaly problem here presented parallels that of the non-existence of invariant polarizations in the Kirillov-Kostant co-adjoint orbits method [29, 19], and the conventional anomaly problem in Quantum Field Theory which manifests itself through the appearance of central charges in the quantum current algebra, absent from the classical (Poisson bracket) algebra [30].

The full reduction of representations in anomalous cases will be achieved by means of a generalized concept of (higher-order) polarization (see Sec. 5). Higher-order polarizations are needed to accommodate the "quantum invariant relations" inside the polarization, and these are given, in general, by operators in the (left) enveloping algebra.

Let us consider a couple of anomalous groups, one of them finite dimensional (the Schrödinger group) and the other one infinite dimensional (the Virasoro group).

The Schrödinger group

To illustrate the Lie algebra structure of an anomalous group, let us first consider the example of the Schrödinger group. This group, or rather the non-extended n -dimensional version of it, was considered in Ref. [19] as an exam-

ple of a group not possessing an *admissible* subalgebra (the equivalent to a full and symplectic polarization in our context). In the simplest 1-dimensional case, G is the semidirect action of the symplectic group $Sp(1, R) \approx SL(2, R)$ on the phase space R^2 . We will consider a central extension of it, the Schrödinger group \tilde{G} , which is given by the semidirect action of $SL(2, R)$ on the Heisenberg-Weyl group. This group includes as subgroups the symmetry group of the free particle, the Galilei Group, as well as the symmetry group of the ordinary harmonic oscillator and the “repulsive” harmonic oscillator (with imaginary frequency), usually known as Newton groups [31]. From the mathematical point of view, it can be obtained from the Galilei (or from either of the Newton) groups by replacing the time subgroup with the three-parameter group $SL(2, R)$. In fact, those kinematical subgroups are associated with different choices of a Hamiltonian inside $SL(2, R)$.

Let us parameterize the Schrödinger group by (x, v, a, t, c, ζ) , where (x, v, ζ) parameterize the Heisenberg-Weyl subgroup ($\zeta \in U(1)$), and (a, t, c) are the parameters for the $SL(2, R)$ subgroup, with $a \in R - \{0\}$, for which we use the following Gauss decomposition [6]:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \quad (49)$$

with $\alpha\delta - \beta\gamma = 1$. For our purposes we only need the Lie algebra (see [6] for a detailed account of the group law and the expressions of vector fields), which is given (in terms of left-invariant vector fields) by:

$$\begin{aligned} \begin{bmatrix} \tilde{X}_a^L, \tilde{X}_t^L \end{bmatrix} &= 2\tilde{X}_t^L & \begin{bmatrix} \tilde{X}_a^L, \tilde{X}_x^L \end{bmatrix} &= \tilde{X}_x^L \\ \begin{bmatrix} \tilde{X}_t^L, \tilde{X}_c^L \end{bmatrix} &= \tilde{X}_a^L & \begin{bmatrix} \tilde{X}_a^L, \tilde{X}_v^L \end{bmatrix} &= -\tilde{X}_v^L \\ \begin{bmatrix} \tilde{X}_a^L, \tilde{X}_c^L \end{bmatrix} &= -2\tilde{X}_c^L & \begin{bmatrix} \tilde{X}_c^L, \tilde{X}_x^L \end{bmatrix} &= -\tilde{X}_v^L \\ \begin{bmatrix} \tilde{X}_t^L, \tilde{X}_x^L \end{bmatrix} &= 0 & \begin{bmatrix} \tilde{X}_c^L, \tilde{X}_v^L \end{bmatrix} &= 0 \\ \begin{bmatrix} \tilde{X}_t^L, \tilde{X}_v^L \end{bmatrix} &= -\tilde{X}_x^L & \begin{bmatrix} \tilde{X}_x^L, \tilde{X}_v^L \end{bmatrix} &= m\tilde{X}_0^L, \end{aligned} \quad (50)$$

Analysing the Lie-algebra 2-cocycle Σ (see Sec. 2.2), we deduce that this central extension is associated with the (exceptional) 2-dimensional orbit of the Schrödinger group, since it contains the entire $SL(2, R)$ subalgebra $\langle \tilde{X}_t^L, \tilde{X}_a^L, \tilde{X}_c^L \rangle$ (the characteristic subalgebra, or isotropy subalgebra of the coadjoint orbit) in its kernel. According to the general scheme, the characteristic subalgebra should enter any full and symplectic polarization

but such a polarization does not exist, and this can be traced back to the fact that the partial complex structure $J = \theta^{Lv} \otimes \tilde{X}_x^L - \theta^{Lx} \otimes \tilde{X}_v^L$ is not preserved by its kernel (the $SL(2, R)$ subalgebra), implying that we cannot project it onto a complex structure on the classical phase-space. We can only find a symplectic, non-full polarization,

$$\mathcal{P} = \langle \tilde{X}_t^L, \tilde{X}_a^L, \tilde{X}_x^L \rangle, \quad (51)$$

and a full, but non-symplectic one,

$$\mathcal{P}_C = \langle \tilde{X}_t^L, \tilde{X}_a^L, \tilde{X}_c^L \rangle \approx SL(2, R). \quad (52)$$

Quantizing with the non-full polarization (51) results in a breakdown of the naively expected correspondence between the operators $\tilde{X}_t^R, \tilde{X}_a^R, \tilde{X}_c^R$ and the basic ones $\tilde{X}_x^R, \tilde{X}_v^R$, i.e. the one suggested by the Noether invariants (see Sec. 2) which can be written as [6]:

$$\begin{aligned} F_t &= -\frac{1}{2m} F_x^2 \\ F_a &= -\frac{1}{m} F_x F_v \\ F_c &= \frac{1}{2m} F_v^2. \end{aligned} \quad (53)$$

This relations characterize the two-dimensional (exceptional) coadjoint orbit, being the *invariant relations* mentioned above.

On the other hand, quantizing with the non-symplectic polarization (52) leads to an unconventional representation in which the wave functions depend on both x and p variables, which contains two irreducible components (see [1]) distinguished by the eigenvalues of the parity operator, which is not in the group. This particular example shows the principal drawback of the higher-order polarization technique, since there are no means of obtaining the parity operator from the group nor its enveloping algebra. As happened with the other polarization, the operators $\tilde{X}_t^R, \tilde{X}_a^R, \tilde{X}_c^R$, neither, are expressed in terms of $\tilde{X}_p^R, \tilde{X}_x^R$.

In both cases the operators $\tilde{X}_t^R, \tilde{X}_c^R$ behave as if they also were basic operators, i.e. as if

$$[\tilde{X}_t^L, \tilde{X}_c^L] = \tilde{X}_a^L + 2k\tilde{X}_0^L \quad (54)$$

would replace the corresponding commutator in (50) with a non-trivial value of the anomaly (pseudo-extension parameter) k . In other words, both quantizations seem to correspond to a four-dimensional (of maximal dimension) orbit, for which there are no "quantum invariant relations" between operators (the quantum counterpart of the invariant relations between Noether invariants).

Therefore, we should start with the Lie algebra (50) with "deformed" commutators like in (54). In this way, we are considering a whole family (this kind of pseudo-extension corresponds to a family of four dimensional orbits associated with the 1-sheet hyperboloid of $SL(2, R)$). To consider the rest of the coadjoint orbits, associated with the 2-sheet hyperboloid or the cone, a different pseudo-extension is required, associated with the compact Cartan subgroup of $SL(2, R)$ of coadjoint orbits, all of them four-dimensional, except for $k = 0$ which is two-dimensional. For all of these pseudoextensions, the non-full polarization (51) is now full and symplectic, since the characteristic subalgebra is smaller (corresponding to the fact that for the four-dimensional orbits the isotropy subalgebra is smaller), $\mathcal{G}_C = \langle \tilde{X}_a^L \rangle$. In the next section we shall see that the fact that there is a full and symplectic polarization does not guarantee the irreducibility of the representation, and that there exist a value of the parameter k for which the representation obtained is reducible, admitting an invariant subspace.

The Virasoro group

Let us comment very briefly on the relevant, although less intuitive, example of the infinite-dimensional Virasoro group. Its Lie algebra can be written as

$$[\tilde{X}_{l_n}^L, \tilde{X}_{l_m}^L] = -i(n - m)\tilde{X}_{l_n+m}^L - \frac{i}{12}(cn^3 - c'n)\Xi, \quad (55)$$

where c parameterizes the central extensions and c' the pseudo-extensions. As is well known, for the particular case in which $\frac{c'}{c} = r^2, r \in \mathbb{N}, r > 1$, the co-adjoint orbits admit no invariant Kählerian structure. In the present approach, this case shows up as an algebraic anomaly. In fact, the characteristic subalgebra is given by $\mathcal{G}_C = \langle \tilde{X}_{l_0}^L, \tilde{X}_{l_{-r}}^L, \tilde{X}_{l_{+r}}^L \rangle$, which is not fully contained in the non-full (but symplectic) polarization $\mathcal{P}^{(r)} = \langle \tilde{X}_{l_{n \leq 0}}^L \rangle$. There also exists a full polarization $\mathcal{P}_C = \langle \tilde{X}_{l_{kr}}^L \rangle, r > 1, k = -1, 0, 1, 2, 3, \dots$ which is not

symplectic since none of the symplectic generators with labels $l_{\pm r'}, r' \neq kr$ are included in the polarization. A detailed description of the representations of the Virasoro group can be found in [20] and references therein.

5 Higher-order Polarizations

In general, to tackle situations like those mentioned above, it is necessary to generalize the notion of polarization. Let us consider the universal enveloping algebra of left-invariant vector fields, $\mathcal{U}\tilde{\mathcal{G}}^L$. We define a **higher-order polarization** \mathcal{P}^{HO} as a maximal subalgebra of $\mathcal{U}\tilde{\mathcal{G}}^L$ with no intersection with the abelian subalgebra of powers of \tilde{X}_0 . With this definition a higher-order polarization contains the maximum number of conditions (expressed in terms of differential operators) compatible with the equivariance condition of the wave functions and with the action of the physical operators (right-invariant vector fields).

We notice that now the vector space of functions annihilated by a higher-order polarization is not, in general, a ring of functions and therefore there is no corresponding foliation; that is, they cannot be characterized by saying that they are constant along submanifolds. If this were the case, it would mean that the higher-order polarization was the enveloping algebra of a first-order polarization and, accordingly, we could consider the submanifolds associated with this polarization. In this sense the concept of higher-order polarization generalizes that of first-order polarization.

The definition of higher-order polarization given above is quite general. In all studied examples higher-order polarizations adopt a more definite structure closely related to given first-order (non-full and/or non-symplectic) ones. According to the until now studied cases, higher-order polarizations can be given a more operative definition: *A higher-order polarization is a maximal subalgebra of $\mathcal{U}\tilde{\mathcal{G}}^L$ the “vector field content” of which is a first-order polarization.* By “vector field content” of a subalgebra \mathcal{A} of $\mathcal{U}\tilde{\mathcal{G}}^L$ we mean the following: Let $V(\mathcal{A})$ be the vector space of complex functions on \tilde{G} defined by

$$V(\mathcal{A}) = \{f \in \mathcal{F}_C(\tilde{G}) / A \cdot f = 0, \forall A \in \mathcal{A}\}. \quad (56)$$

Now we consider the ring $R(\mathcal{A})$ generated by elements of $V(\mathcal{A})$. With $R(\mathcal{A})$

we associate the set of left-invariant vector fields defined by

$$L_{\tilde{X}^L} h = 0, \quad \forall h \in R(\mathcal{A}). \quad (57)$$

This set of left-invariant vector fields is a Lie subalgebra of $\tilde{\mathcal{G}}^L$ and defines the vector field content of \mathcal{A} , which proves to be a first-order polarization.

Even though a higher-order polarization contains the maximal number of conditions (expressible in terms of differential equations) there can be non-trivial operators acting on the Hilbert space of wave functions which are not differential, and therefore are not contained in the enveloping algebra. This implies that higher-order polarizations will not guarantee, in general, the irreducibility of the resulting representation, since there can be non-trivial and non-differential operators acting on the resulting Hilbert space and commuting with the representation.

A simple example suggesting the need of a generalization of the concept of higher-order polarization corresponds to the non-irreducible representation associated with the non-symplectic polarization (52) of the Schrödinger group. This representation cannot be further reduced by enlarging (52) to a higher-order polarization \mathcal{P}^{HO} . A full reduction requires the inclusion in \mathcal{P}^{HO} of the parity operator commuting with the representation. The generalization of the concept of higher-order polarization so as to include this kind of operators not reachable by the enveloping algebra of the group, as well as a constructive characterization of those operators deserves a separate study. We outline a possible way to deal with the problem. We consider the group of unitary automorphisms A of the space \mathcal{F} of complex valued functions on \tilde{G} satisfying the equivariance condition (15) (or, rather, of the completion of it in the scalar product defined by the left invariant Haar measure). This (huge) group A will replace the role of the enveloping algebra, and the role of the higher-order polarization will be played by a *polarizing subgroup* P of A . This polarizing subgroup P must be a maximal subgroup of A satisfying the following conditions:

- i) $P \cap U(1) = \{I\}$, where $U(1)$ is the vertical subgroup.
- ii) If $T \subset A$ is the left regular representation of \tilde{G} restricted to the space of equivariant functions, then T must be in the normalizer of P , i.e. $[P, T] \subset P$.

The reduced space \mathcal{H} is defined as the set of equivariant functions on which all the elements of P act as the identity. Then T , when restricted to \mathcal{H} , is irreducible. To see it, suppose that B is an invariant subspace of \mathcal{H} , and p the (self-adjoint) projector on it, which commutes with T . Then $e^{i\lambda p} = I + (e^{i\lambda} - 1)p$ is a unitary operator on \mathcal{H} commuting with T . It can be trivially extended to a unitary operator on \mathcal{F} commuting with T , and therefore, since P is maximal, it must be contained in P . But P acts trivially on \mathcal{H} , and therefore p annihilates \mathcal{H} . Thus, B is trivial and T is irreducible.

Let P' be the minimal subgroup of P determining the same Hilbert space \mathcal{H} satisfying the condition $P'\Psi = \Psi$. Let P'_0 be the connected component of P' , and \mathcal{P}' its Lie algebra, which, according to the Stone-von Neumann theorem, will be constituted by self-adjoint operators in (some dense domain of) \mathcal{F} . Clearly, the condition $\mathcal{P}'\Psi = 0$ is equivalent to the condition $P'\Psi = \Psi$ and may determine the same Hilbert space \mathcal{H} only if P' is connected, i.e. if P'/P'_0 is trivial.

For the cases in which a first-order (full and symplectic) polarization \mathcal{P}^1 exists, and it is enough to reduce completely the representation T , it is clear that \mathcal{P}' will coincide with \mathcal{P}^1 and will be constituted by first order differential operators.

Also, for the cases in which a higher-order polarization \mathcal{P}^{HO} is enough to obtain an irreducible representation, \mathcal{P}' will coincide with \mathcal{P}^{HO} , and will be constituted by higher-order differential operators.

Those cases, like the example commented above of the Schrödinger group with the polarization (52), in which a higher-order polarization is not enough to obtain an irreducible representation, lie in the category of groups for which \mathcal{P}' contains non-differential self-adjoint operators or P' is not connected (or both of them). The example of the Schrödinger group with the polarization (52) lies in the second category, since the operator we need to completely reduce the representation is the parity operator, which is discrete, and therefore belongs to P'/P'_0 .

To see how a higher-order polarization operates in practice, we shall consider first a simple non-anomalous example like the Harmonic Oscillator in configuration space, and later we will come back to the cases of the Schrödinger and Virasoro groups.

The Harmonic Oscillator

The (quantum) Harmonic Oscillator group (we shall restrict to the one-dimensional case, since it presents all the interesting features and the treatment is far simpler), as the Galilei group, is a semidirect product of the time translations and the Heisenberg-Weyl group. The difference relies precisely on the semidirect actions, which correspond to different choices for uniparametric subgroups of $Sp(2, R) \approx SL(2, R)$ acting on H-W as linear canonical transformations. For the case of the Galilei group, the time translations are those generated by \hat{P}^2 , and constitutes a non-compact subgroup, while for the harmonic oscillator group, time translations are generated by $\hat{P}^2 + \hat{X}^2$, which corresponds to the compact subgroup $SO(2)$ of $SL(2, R)$.

The harmonic oscillator group possesses nontrivial group cohomology, but the pseudo-cohomology is trivial (although pseudo-extensions can be introduced, all of them lead to equivalent representations).

The group law for the harmonic oscillator group can be obtained from this semidirect action (in fact, it can be seen as a central extension of the Euclidean group $E(2)$), see [6]:

$$\begin{aligned}
t'' &= t' + t \\
x'' &= x + x' \cos \omega t + \frac{p'}{m\omega} \sin \omega t \\
p'' &= p + p' \cos \omega t - m\omega x' \sin \omega t \\
\zeta'' &= \zeta' \zeta e^{\frac{i}{2\hbar}(x' p \cos \omega t - p' x \cos \omega t + (\frac{p' p}{m\omega} + \omega x' x) \sin \omega t)} .
\end{aligned} \tag{58}$$

It is easy to see that under the limit $\omega \rightarrow 0$ (which corresponds to a group contraction in the sense of Inönü and Wigner) we obtain the group law for the Galilei group.

The left-invariant vector fields are:

$$\begin{aligned}
\tilde{X}_t^L &= \frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial x} - m\omega^2 x \frac{\partial}{\partial p} \\
\tilde{X}_x^L &= \frac{\partial}{\partial x} - \frac{p}{2\hbar} \tilde{X}_0 \\
\tilde{X}_p^L &= \frac{\partial}{\partial p} - \frac{x}{2\hbar} \tilde{X}_0 \\
\tilde{X}_\zeta^L &= \frac{\partial}{\partial \phi} \equiv \tilde{X}_0 ,
\end{aligned} \tag{59}$$

and the right ones are:

$$\begin{aligned}
\tilde{X}_t^R &= \frac{\partial}{\partial t} \\
\tilde{X}_x^R &= \cos \omega t \frac{\partial}{\partial x} - m\omega \sin \omega t \frac{\partial}{\partial p} + \frac{1}{2\hbar} (p \cos \omega t + m\omega x \sin \omega t) \tilde{X}_0 \\
\tilde{X}_p^R &= \cos \omega t \frac{\partial}{\partial p} + \frac{1}{m\omega} \sin \omega t \frac{\partial}{\partial x} - \frac{1}{2\hbar} (x \cos \omega t - \frac{p}{m\omega} \sin \omega t) \tilde{X}_0 \\
\tilde{X}_\zeta^R &= \frac{\partial}{\partial \phi} \equiv \tilde{X}_0.
\end{aligned} \tag{60}$$

The commutation relations for theses vector fields are:

$$\begin{aligned}
[\tilde{X}_t^R, \tilde{X}_x^R] &= -m\omega^2 \tilde{X}_p^R \\
[\tilde{X}_t^R, \tilde{X}_p^R] &= \frac{1}{m} \tilde{X}_x^R \\
[\tilde{X}_x^R, \tilde{X}_p^R] &= -\frac{1}{\hbar} \tilde{X}_0.
\end{aligned} \tag{61}$$

The quantization 1-form Θ (we redefine it with a factor \hbar) is:

$$\Theta = \hbar \frac{d\zeta}{i\zeta} + \frac{1}{2} (pdx - xdp) - \left(\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \right) dt, \tag{62}$$

and the characteristic subalgebra is $\mathcal{G}_C = \langle \tilde{X}_t^L \rangle$. The partial complex structure is given by $J = \theta^{Lp} \otimes \tilde{X}_x^L - \theta^{Lx} \otimes \tilde{X}_p^L$. If we look for first order polarizations, we find that we are not able to find a full and symplectic real polarization (in some sense, the Harmonic Oscillator is an anomalous system, see [32]), only complex polarizations (making use of the natural complex structure of the phase space $R^2 \approx C$ of the system, induced again by J) can be full and symplectic. They are of the form:

$$\mathcal{P}_\pm = \langle \tilde{X}_t^L, \tilde{X}_x^L \pm im\omega \tilde{X}_p^L \rangle, \tag{63}$$

and lead to the Bargmann-Fock representation of the harmonic oscillator in terms of (anti-)holomorphic functions (see [23, 33]).

But now we could be interested in using only real polarizations for obtaining the configuration (or momentum) space representation, and this can

only be achieved if we resort to higher-order polarizations. For this simple case it is an easy task to obtain the higher-order polarizations, since we only have to consider the subalgebras of the left-enveloping algebra generated by the Casimir

$$\tilde{X}_t^L - \frac{i\hbar}{2m} (\tilde{X}_x^L)^2 - \frac{i\hbar m\omega}{2} (\tilde{X}_p^L)^2, \quad (64)$$

and \tilde{X}_p^L or \tilde{X}_x^L . The subalgebra generated by the Casimir and \tilde{X}_t^L is not maximal, since we can still add $\tilde{X}_x^L \pm im\omega \tilde{X}_p^L$. Therefore, there are essentially two real higher-order polarizations, $\mathcal{P}_x^{HO} = \langle \tilde{X}_t^L - \frac{i\hbar}{2m} (\tilde{X}_x^L)^2, \tilde{X}_p^L \rangle$, which leads to the representation in configuration space, and $\mathcal{P}_p^{HO} = \langle \tilde{X}_t^L - \frac{i\hbar m\omega}{2} (\tilde{X}_p^L)^2, \tilde{X}_x^L \rangle$, leading to the representation in momentum space. These two representations are unitarily equivalent, the unitary transformation being the Fourier transform, and are also unitarily equivalent to the Bargmann-Fock representation through the Bargmann transform (see the comments on the case of the abelian group R^k).

Let us consider, for instance, the polarization \mathcal{P}_x^{HO} leading to configuration space. The solutions to the polarization equations are:

$$\begin{aligned} \tilde{X}_p^L \Psi = 0 & \rightarrow \Psi = \zeta e^{-\frac{i}{\hbar} \frac{px}{2}} \Phi(x, t) \\ (\tilde{X}_t^L - \frac{i\hbar}{2m} \tilde{X}_x^L \tilde{X}_x^L) \Psi = 0 & \rightarrow i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\partial^2 \Phi}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 \Phi. \end{aligned} \quad (65)$$

The last equation is the well-know Schrödinger equation for the harmonic oscillator in configuration space, with the standard solutions in terms of Hermite Polynomials:

$$\begin{aligned} \Phi \equiv \sum_{n=0}^{\infty} c_n \Phi_n(x, t) &= \sqrt{\frac{\omega}{2\pi}} \left(\frac{m\omega}{\hbar\pi} \right)^{1/4} \\ &\times \sum_{n=0}^{\infty} \frac{c_n}{2^{n/2} \sqrt{n!}} e^{-\frac{m\omega}{2\hbar} x^2} e^{-i(n+1/2)\omega t} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right). \end{aligned} \quad (66)$$

The scalar product can be obtained from the (left-invariant) Haar measure on the group. Indeed, $i_{\tilde{X}_p^L} \Omega^L = dx \wedge dt$ is an invariant measure on the quotient space G/G_p , with G_p the subgroup generated by \tilde{X}_p^L :

$$\langle \Psi' | \Psi \rangle = \int dx dt \Phi'(x, t)^* \Phi(x, t). \quad (67)$$

We would like to stress that although the representations here obtained in configuration space (or its analogous in momentum space) are unitarily equivalent to the one obtained in the Bargmann-Fock space, the latest requires of a process of "unitarization" to obtain the correct energy of the vacuum $E_0 = 1/2\hbar\omega$, which otherwise would be zero (see [23]). In the literature this problem was solved recurring to the "metaplectic correction" (see [4]), and here we obtain the correct result resorting only to higher-order polarizations. Moreover, this fact can be seen as a reminiscence of the anomaly of the Schrödinger group, which causes the correct ordering of the operators.

The Schrödinger group

Now we consider the case of the Schrödinger group and the representation associated with the non-full polarization (51) for which the operators $\tilde{X}_t^R, \tilde{X}_c^R$ are basic. As stated before, for commutation relations like (54), the polarization (51) becomes full and symplectic, as far as $k \neq 0$. Thus one would think that the associated representations (with two degrees of freedom) are irreducible. However, for a particular value (the quantum value of the anomaly) $k = \frac{1}{4}$, the representation of the Schrödinger group *becomes reducible* and, *on the invariant subspace*, the operators $\tilde{X}_t^R, \tilde{X}_c^R$ do really express as $\hat{p}^2/2m, \hat{x}^2/2$, respectively. The invariant subspace is constituted by the solutions of a second-order polarization which exists only for $k = \frac{1}{4}$:

$$\mathcal{P}^{HO} = \langle \tilde{X}_t^L, \tilde{X}_a^L, \tilde{X}_x^L, \tilde{X}_c^L - \frac{i}{2m}(\tilde{X}_v^L)^2 \rangle, \quad (68)$$

This result indicates that for the (four-dimensional) coadjoint orbit associated with the value $k = \frac{1}{4}$ of the pseudo-extension parameter, although no classical invariant relations like (53) exist, there exist "quantum invariant relations", in such a way that the quantum system possesses only one degree of freedom (see [34] for a detailed discussion on this question). The quantum invariant relations are of the form:

$$\begin{aligned} \tilde{X}_t^{LHO}\Psi &= 0 \\ \tilde{X}_a^{LHO}\Psi &= 0 \\ \tilde{X}_c^{LHO}\Psi &= 0, \end{aligned} \quad (69)$$

where the second-order operators are given by

$$\tilde{X}_t^{LHO} = \tilde{X}_t^L - \frac{i}{2m}(\tilde{X}_x^L)^2$$

$$\begin{aligned}
\tilde{X}_a^{LHO} &= \tilde{X}_a^L - \frac{i}{m} \tilde{X}_v^L \tilde{X}_x^L \\
\tilde{X}_c^{LHO} &= \tilde{X}_c^L + \frac{i}{2m} (\tilde{X}_v^L)^2.
\end{aligned} \tag{70}$$

In fact, note that the polarization \mathcal{P}^{HO} is equivalent to the one given by:

$$< \tilde{X}_t^{LHO}, \tilde{X}_a^{LHO}, \tilde{X}_c^{LHO}, \tilde{X}_x^L >, \tag{71}$$

and k must be $k = \frac{1}{4}$ for this to be a higher-order polarization, or, in other words, for (69) to constitute true "quantum invariant relations". The fact that classical and quantum invariant relations are realized for different values of k ($k = 0$ and $k = \frac{1}{4}$, respectively) can be thought of as being due to normal ordering problems, in the operator \tilde{X}_a^{LHO} to be precise.

It is worth noting that in the polarization (71), the characteristic subalgebra which, for $k = 0$, is given by $\mathcal{G}_C = < \tilde{X}_t^L, \tilde{X}_a^L, \tilde{X}_c^L > \approx SL(2, R)$, has been substituted by a higher-order characteristic subalgebra $\mathcal{G}_C^{HO} = < \tilde{X}_t^{LHO}, \tilde{X}_a^{LHO}, \tilde{X}_c^{LHO} >$, which, for $k = \frac{1}{4}$, satisfy identical commutation relations but commute with the generators \tilde{X}_x^L and \tilde{X}_v^L , and therefore can be included in a polarization (although of higher-order type).

Physical applications of this particular representation, although in the harmonic oscillator realization are found in Quantum Optics [35] although no reference to the connection between anomalies and the restriction of k has been made. Note that when restricted to the $SL(2, R)$ subgroup, the representation obtained is reducible decomposing in two irreducible ones with Bargmann indices $k = \frac{1}{4}$ and $k + \frac{1}{2} = \frac{3}{4}$.

The Virasoro group

In a similar way, in the case of the Virasoro group, for particular values of the parameters c, c' or equivalently $c, h \equiv \frac{c-c'}{24}$ given by the Kac formula [36], the "quantum values" of the anomaly, the representations given by the first order non-full (symplectic) polarizations are reducible since there exist invariant subspaces characterized by certain higher-order polarization equations [20], which constitute the "quantum invariant relations". Note that there is no one-to-one correspondence between the values of c'/c characterizing the coadjoint orbits of the Virasoro group (the classical values of the anomaly) and the values allowed by the Kac formula (the quantum values of

the anomaly), a fact which must be interpreted as a breakdown of the notion of classical limit.

6 Comments and outlooks

Let us comment further on the relationship between the present formalism and the more conventional method formulated on the co-adjoint orbits in \mathcal{G}^* . We recall that if we denote by $\mu_\Theta \in \tilde{\mathcal{G}}^*$ the element we get by evaluating Θ at the origin of the group \tilde{G} , we obtain a symplectic orbit in $\tilde{\mathcal{G}}^*$ passing through μ_Θ . This orbit is diffeomorphic to $\tilde{G}/\text{Ker } d\Theta$. Therefore, we replace the study of symplectic orbits in $\tilde{\mathcal{G}}^*$ with the study of some quotient spaces in \tilde{G} . At this point, however, instead of looking for canonical co-ordinates for ω_{μ_Θ} or Θ , which in general do not exist globally, we use left-invariant vector fields in \tilde{G} , which are in the polarization, to select an irreducible subspace of functions. These vector however, do not project, in general, onto the quotient space, thus they are not canonically available in the analysis in terms of symplectic coadjoint orbit through μ_Θ .

Another important advantage of the present approach is that on $\tilde{\mathcal{G}}^*$ the enveloping algebra of $\tilde{\mathcal{G}}$ is traded with polynomial functions and, therefore, we can only deal with their associated vector fields via the Poisson bracket on $\tilde{\mathcal{G}}^*$, i.e. they will be first order. This is due to the fact that Witt correspondence (see e.g. [37]) from the universal enveloping algebra to polynomials on $\tilde{\mathcal{G}}^*$ is only a vector-space map, as it destroys the algebra character.

More interesting and subtle is perhaps the comparison with the Borel-Weyl-Bott group representation technique [17] intended for finite-dimensional semisimple groups. There, the starting point is a principal fibration of a semisimple group G on the quotient G/H of G by the Cartan subgroup H , and then a condition analogous to the polarization condition is imposed by means of the generators in a Borel subalgebra constituted by the generators of H as well as those associated with a maximal set of positive roots. The notion of Borel subalgebra coincides with our definition of polarization for the case of finite-dimensional semisimple groups, for which polarizations are always full and symplectic (that is, finite-dimensional semisimple groups are not anomalous). Apart from the obvious similarity, there are non-trivial differences. The BWB mechanism does not apply to the infinite-dimensional case, where the Whitehead Lemma no longer holds and non-trivial coho-

mology appears characterizing projective representations, as is the case of Kac-Moody and Virasoro groups. Furthermore, these groups can be anomalous, so that Borel-like subalgebras do not exist. The more representative example is constituted by the Virasoro group (see Sec. 4).

One should also add that working on \tilde{G} allows us to use the left and right universal enveloping algebras. But now we can use ideas from quantum groups to consider higher-order polarizations in deformations of these enveloping algebras. The main feature of their null mutual commutator is still preserved, so that we could have unitary representations of the deformed right-invariant enveloping algebras. This procedure allows us to tackle the problem of quantization of Lie-Poisson groups without going through the “star product quantization”. We shall take up these aspects somewhere else.

As far as the geometry of anomalous systems is concerned, we want to remark that, as commented before, the use of higher-order polarizations does not lead, in general, to the notion of Lagrange submanifold associated with the representation. Higher-order polarizations are subalgebras of the left enveloping algebra of \tilde{G} which are not necessarily the enveloping algebra of a given subalgebra of \tilde{G} , so that the set of solutions of the polarization equations and equivariance condition, is a subspace $\mathcal{H} \subset \mathcal{F}^C(\tilde{G})$ which is not necessarily a subalgebra. Then, the Gelfan’-d-Kolmogoroff theorem [38] cannot be applied to identify a submanifold of \tilde{G} with the set $Hom_C(\mathcal{H}, C)$. From the point of view of Quantum Mechanics (resp. group representation) the lack of classical integrability of polarizations makes unclear the idea of classical limit and the association of specific phase spaces (resp. co-adjoint orbits) with actual quantizations (resp. irreducible unitary representations). This fact was first stated when studying the irreducible representations of the Virasoro group “associated” with the non-Kähler orbits $diff S^1/SL^{(r)}(2, R)$, $r > 1$ [39, 20]. More specifically, when the characteristic subalgebra in a higher-order polarization is itself of higher order, \mathcal{G}_C^{HO} , i.e. there are elements in \mathcal{P}^{HO} which never reproduce X_0 , nor any power of it, under commutation with the whole enveloping algebra, the exponential of \mathcal{G}_C^{HO} is not a subgroup of \tilde{G} and therefore the quotient G/\mathcal{G}_C^{HO} is not defined in general, unlike the non-anomalous case where G/\mathcal{G}_C , the quotient of G by the integrable distribution \mathcal{G} , constitutes the classical phase space. Furthermore, the generalized equations of motion of higher-order type select wave functions on which the group \tilde{G} acts through a representation in terms of higher-order differential operators. If the group is not anomalous, this representation will be unitarily

equivalent to a representation obtained by means of a first-order polarization, as it happens for the Galilei group or the harmonic oscillator group, here considered, where there exist a unitary operator relating the representations in configuration space in terms of higher-order differential operators with the ones in momentum space or Bargmann-Fock space, respectively, in terms of first-order differential operators. But if the group is anomalous, there can be representations obtained by means of higher-order polarizations that are not unitarily equivalent to any one in terms of first-order differential operators.

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